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Characteristics of the unsteady motion on transversely sheared mean flows

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In this paper we obtain an explicit representation for the unsteady motion on a transversely sheared mean flow that corresponds to the gustline motion on a uniform mean flow. The important features of this motion are discussed. It is shown that its velocity, pressure and vorticity are all induced by a certain disturbance field that is a linear combination of the vorticity and particle-displacement fields and is everywhere frozen in the mean flow. The general ideas are illustrated by considering the scattering of a gust by a half-plane embedded in a shear flow.

1. Introduction

In an inviscid non-heat-conducting compressible fluid the small amplitude unsteady motion about a steady mean flow is governed by the linearized gasdynamic equations. The general character of this motion is well understood in the case where the steady mean flow is uniform, i.e. where the mean velocity is a constant (see Carrier & Carlson 1946; Kovásznay 1953; Chu & Kovásznay 1958). In these flows the unsteady motion is simply the sum of two disturbances: (i) a frozen (i.e. purely convected) disturbance or mode that has zero divergence (i.e. it is entirely vortical), has no effect on the pressure fluctuations (apart from the coupling that occurs at any solid boundaries that may be present) and is often called a gust in unsteady airfoil theory; (ii) an irrotational disturbance that is directly related to the pressure fluctuations and is, as a result, connected with any acoustic-type motion that may occur. We therefore refer to the latter disturbance as acoustic though we realize that it will occur even when the fluid is incompressible. Since these two modes of motion are coupled only at the boundaries of the flow, each of them must, by itself, be a solution to the linearized gasdynamic equations. (A simple proof of these results is given on pp. 220 and 221 of Goldstein 1976.)

The next simplest situation is when the steady mean motion is a transversely sheared flow. Here the mean velocity has the same direction at every point of the flow but its magnitude can vary from point to point in any plane that is perpendicular to this direction. Such a flow will satisfy the inviscid non-heat-conducting equations of motion if we require that the pressure be everywhere constant and that the density remains constant along the surfaces of constant mean velocity. (See pp. 6–10 of Goldstein (1976) for details. The most familiar examples of such flows are the two-dimensional shear flows that are studied in the theory of hydrodynamic stability.)

The character of the unsteady motion on transversely sheared flows is considerably more complex than it is when the mean flow is uniform. Here the pressure and vorticity fluctuations are coupled and there are no velocity fluctuations that are convected with the mean flow. In fact, even the unsteady vorticity perturbation does not have this property. Part of the purpose of this paper is to sort out the nature of these flows.

We recognize at the outset that we cannot hope to decompose the unsteady motion into acoustical and vortical parts at all points of an arbitrary transversely sheared mean flow. However, the major usefulness of this decomposition is connected with the formulation of boundary-value problems associated with the 'scattering' of one of these types of motion into another by solid surfaces or other types of boundary placed within a uniform mean flow. In problems of this type we are frequently interested in flow fields that extend to infinity in all directions, in which case it is sufficient to be able to distinguish the vortical motion far upstream from the region containing the scattering surfaces.

It will be shown subsequently that the linearized equations for a transversely sheared mean flow possess a solution whose vorticity field is frozen in the flow far upstream and, in fact, exhibits a number of other characteristics of the uniform flow vortical solution in the rest of the flow. We should like to show that this solution is the natural generalization of the uniform mean flow gust solution. This can be done if we can extract certain properties of the uniform flow gust solution which are just sufficient to determine this quantity uniquely and which are also possessed by and are just sufficient to determine uniquely the transversely sheared flow solution alluded to above. However, the decomposition of the unsteady motion on a uniform flow into acoustical and vortical parts is not unique. This is because the linearized equations possess solutions that have the properties of both the acoustic and the vortical solution. But (as shown in § 2) there will be only one vortical solution which will be both bounded (i.e. finite) at infinity and continuous at all points of space and this solution will be uniquely determined by the upstream vorticity distribution. Since we usually think of the incident gust or vortical mode in a scattering problem as the portion of the motion that would exist if no scattering surfaces were present, it is reasonable to define the vortical component of the solution in a unique fashion by adding the slight additional requirement that it be everywhere continuous and bounded.

In § 2 we show that the uniform mean flow vortical solution described above can be uniquely characterized by a certain minimum wavenumber bandwidth property. In § 3 we construct a solution to the linearized gasdynamic equations for a transversely sheared mean flow that is also uniquely characterized by this minimum wavenumber bandwidth property. We therefore call this solution a gust. It turns out that it is composed entirely of waves that move downstream in the mean flow direction with velocities lying between the maximum and minimum flow velocity and as in the uniform mean flow case is uniquely determined by the upstream vorticity distribution.

The general properties of the gust are discussed in § 3.3. It is shown that the complete solution to any given problem involving the scattering of an incident vorticity field will be the sum of two solutions, one of which (the gust) is uniquely determined by the upstream vorticity distribution and is independent of any solid surfaces and acoustic sources that may be in the flow. The remaining portion will then arise from the scattering or reflexion of the gust by any solid surfaces that may be present or from any acoustic sources or incident acoustic waves (which, as we shall show, can be distinguished from the vortical motion far upstream) that may be imposed on the flow. The latter solution can therefore be referred to as the acoustic portion.

It is also shown in § 3.3 that the pressure, velocity and vorticity fields of the gust can be thought of as being induced by a certain disturbance field that is everywhere frozen in the flow and convected downstream at the local mean flow velocity. This quantity is shown in § 5 to be a simple linear combination of the vorticity, pressure and particle-displacement fields.

The limiting behaviour of the gust solution as the mean flow velocity becomes everywhere constant is discussed in § 4. It is shown that the expected reduction to the constant mean flow vortical mode behaviour (described above) is indeed achieved. However, the frozen vorticity in the mainstream does not, in general, approach the imposed upstream vorticity distribution as the streamwise co-ordinate x goes to infinity in the upstream (i.e. negative) direction unless the constant mean flow limit is approached in a certain fashion. This non-uniform behaviour is shown to result from the fact that the limit $x \rightarrow -\infty$ cannot be interchanged with the limiting operation in which the mean flow Mach number distribution becomes constant. Such non-uniform limits are, of course, quite common in fluid mechanics, the best known of these being the one associated with the boundary layer, which arises because the limit of infinite Reynolds number cannot be interchanged with the limit where the normal surface co-ordinate approaches zero (Cole 1968, pp. 142ff.).

In § 6 the general ideas are used to study the scattering of a gust by a semi-infinite plate in a non-uniform mean flow. Simple formulae for the far-field pressure fluctuations are obtained in the long-wavelength limit (i.e. when the wavelength of the gust is long compared with the transverse extent of the mean flow). The convective effects are explicitly exhibited in terms of Doppler factors. The directivity patterns are compared with those resulting from an approximate calculation based on a conventional free-space zero mean flow dipole model.

2. The gust solution on a uniform mean flow

In this section we construct the vortical solution for the small amplitude unsteady motion on a uniform (i.e. constant velocity) mean flow. We then show how this result can be uniquely characterized in a way that can be generalized to the case of a transversely sheared mean flow.

As indicated in the introduction, we suppose that the flow extends to infinity in all directions and, in order to simplify the presentation, we suppose that the motion is two-dimensional. The properties of the gust or vortical solution given in the introduction imply that its pressure p_g and x and y velocity components u_g and v_g, respectively, must satisfy

$$\left.\begin{aligned} \left(\frac{\partial}{\partial t} + c_0 M \frac{\partial}{\partial x}\right)\begin{pmatrix} u_g \\ v_g \end{pmatrix} = 0, &\quad \text{frozen disturbance condition,} \\ \partial u_g/\partial x + \partial v_g/\partial y = 0, &\quad \text{zero-divergence condition,} \\ p_g = 0, &\quad \text{constant pressure condition,} \end{aligned}\right\} \tag{2.1}$$

where c_0 is the mean speed of sound, M is the constant mean flow Mach number, x

denotes the co-ordinate in the mean flow direction and y denotes a co-ordinate transverse to this direction. The associated vorticity is

$$\omega_g = \partial v_g/\partial x - \partial u_g/\partial y.$$

We are, for the most part, interested in motions that persist for all time and not in the effects of initial transients or instabilities of the flow that grow with time. Thus without loss of generality we can restrict our attention to the case where the unsteady motion has harmonic time dependence, so that the fluctuations in pressure, vorticity and transverse and axial velocity are of the form

$$\left.\begin{aligned} p_g &= \bar{p}_g(x,y)\,e^{-i\omega t}, \quad \omega_g = \bar{\omega}_g(x,y)\,e^{-i\omega t}, \\ v_g &= \bar{v}_g(x,y)\,e^{-i\omega t}, \quad u_g = \bar{u}_g(x,y)\,e^{-i\omega t}, \end{aligned}\right\} \tag{2.2}$$

respectively.

It is easy to show that the most general harmonic solution to (2.1) that is continuous at all points is given by

$$\bar{p}_g \equiv 0, \tag{2.3}$$

$$\bar{v}_g = \tilde{V}(y)\,e^{ikx/M}, \quad \bar{u}_g = \tilde{U}(y)\,e^{ikx/M}, \tag{2.4a, b}$$

$$\bar{\omega}_g = \Omega(y)\,e^{ikx/M}, \tag{2.5}$$

where

$$\tilde{U}(y) = \frac{iM}{k}\frac{d\tilde{V}(y)}{dy}, \tag{2.6}$$

$$\tilde{V}(y) \equiv -\frac{i}{2}\left[e^{ky/M}\int_y^\infty e^{-k\eta/M}\,\Omega(\eta)\,d\eta + e^{-ky/M}\int_{-\infty}^y e^{k\eta/M}\,\Omega(\eta)\,d\eta\right] + C_1\,e^{ky/M} + C_2\,e^{-ky/M},$$

$k \equiv \omega/c_0$, $\Omega(y)$ can be any continuous function of y, and C_1 and C_2 are arbitrary constants. But this solution will be bounded (i.e. finite) at infinity only if $\Omega(y)$ is bounded at infinity and $C_1 = C_2 = 0$. Consequently,

$$\tilde{V}(y) \equiv -\frac{i}{2}\left[e^{ky/M}\int_y^\infty e^{-k\eta/M}\,\Omega(\eta)\,d\eta + e^{-ky/M}\int_{-\infty}^y e^{k\eta/M}\,\Omega(\eta)\,d\eta\right]. \tag{2.7}$$

Thus when the motion is harmonic, the most general continuous bounded vortical solution for the unsteady motion on a uniform flow is given by (2.2)–(2.7). It is determined everywhere in the flow by the upstream transverse vorticity distribution $\Omega(y)$ and has the form of a wave travelling in the mean flow direction with a propagation speed equal to the mean flow velocity $c_0 M$. *It therefore has the single axial wavenumber k/M for each frequency ω.* The remaining portion of the solution to the linearized gasdynamic equations, i.e. the acoustic part, will contain axial wavenumbers lying between plus and minus infinity. Since this latter portion is irrotational, it can be added to the vortical solution without altering the upstream vorticity distribution. Thus there are many solutions that are consistent with any imposed upstream vorticity field but the vortical solution is distinguished from the rest of these by virtue of its being the continuous bounded solution that has a smaller axial wavenumber bandwidth than any other such solution.

The solution for a general time-dependent motion (of the type alluded to above) can, of course, be obtained by superposing solutions of type (2.2) covering all the frequencies that comprise the motion. The gust (i.e. vortical) solution is still uniquely characterized as *the everywhere continuous bounded solution that will have the smallest* *possible bandwidth of axial wavenumbers at any given frequency and still produce a prescribed upstream vorticity distribution.* In the next section we show that this characterization can be extended to transversely sheared mean flows.

Equation (2.7) can also be written as

$$\tilde{V}(y) = \frac{1}{2i}\int_{-\infty}^{\infty} \exp(-k|y-\eta|/M)\,\Omega(\eta)\,d\eta.$$

This result taken togethei with (2.2) and (2.4)–(2.6) shows that the velocity field of the gust is induced by the transverse distribution $\Omega(y)\exp[i(kx/M-\omega t)]$ of convected disturbances that comprise the vorticity field and that the effect of each of these disturbances is diminished by an exponential damping (or transmission) factor $\exp(-k|y-\eta|)$ that depends only on the transverse distance between the elemental vorticity disturbance and the point where the velocity is observed.

3. Gust solution for transversely sheared mean flows

In order to simplify the presentation we restrict our attention to constant-density parallel shear flows, i.e. to mean flows that vary in a direction perpendicular to a plane (say the x, z plane). However, most of the results will apply with more or less obvious modifications to a general transversely sheared mean flow.

As in the previous section, we suppose that the motion is two-dimensional and for the reasons given in that section we restrict our attention to the case where the unsteady motion has harmonic time dependence. Then the fluctuations in pressure and transverse and axial velocity are of the form

$$p = \overline{p}(x,y)\,e^{-i\omega t}, \quad v = \overline{v}(x,y)\,e^{-i\omega t}, \quad u = \overline{u}(x,y)\,e^{-i\omega t},$$

where x still denotes the co-ordinate aligned with the mean flow, which we can suppose without loss of generality (by subjecting the problem to a Galilean transform if necessary) to be always in the $+x$ direction, while y is the transverse co-ordinate in whose direction the mean velocity variation takes place. It is shown in books on stability that the axial Fourier transforms of these quantities (which are appropriate here because the flow is assumed to extend to infinity)

$$\left.\begin{aligned} P(\alpha,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\alpha x}\overline{p}(x,y)\,dx, \\ V(\alpha,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\alpha x}\overline{v}(x,y)\,dx \\ \text{and}\qquad U(\alpha,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\alpha x}\,\overline{u}(x,y)\,dx \end{aligned}\right\} \tag{3.1}$$

satisfy the coupled set of first-order ordinary differential equations (Betchov & Criminale 1967, p. 176)

$$P' = i\rho_0 c_0(k-\alpha M)\,V, \tag{3.2a}$$

$$\alpha P = \rho_0 c_0[(k-\alpha M)\,U + iM'V], \tag{3.2b}$$

$$(k-\alpha M)\,P = \rho_0 c_0(\alpha U - iV'), \tag{3.2c}$$

where the primes denote differentiation with respect to y, ρ_0 and c_0 are the mean density and speed of sound, $M \equiv M(y)$ is the mean flow Mach number and as before we have put $k \equiv \omega/c_0$. It is well known from the theory of hydrodynamic stability of compressible flows (see, for example, Betchov & Criminale 1967, pp. 175–177) that any two of the dependent variables can be eliminated from these equations to obtain a single second-order equation for the remaining variable. Each of the resulting equations possesses two linearly independent solutions, which we shall denote by using subscripts on the letters P, U and V. Of course, the solutions to any two different equations are not independent of one another but rather are connected through (2.2). Thus any given solution to the equation for P will generate a unique solution to the equation for V and a unique solution to the equation for U. We denote such triples of corresponding solutions by using the same subscript on the three letters P, U and V.

3.1. *Properties of the solutions to* (3.2)

The general character of the solutions to (3.2) is easily deduced from the theory of ordinary differential equations and has already been studied in connexion with the theory of hydrodynamic stability. We shall state here certain of their properties that will be needed in the following development. Thus let y_0 denote the point where $M(y) = k/\alpha$. Then if $M(y)$ is not constant (3.2*a–c*) possess one solution, say

$$\mathbf{Z}_1(\alpha, y) \equiv \{P_1(\alpha, y), V_1(\alpha, y), U_1(\alpha, y)\}, \tag{3.3}$$

such that

$$\left.\begin{aligned} P_1(\alpha, y) &= O((\alpha M - k)^3) = O((y-y_0)^3) \qquad &(3.4a)\\ V_1(\alpha, y) &= \frac{P_1'}{i\rho_0 c_0(k-\alpha M)} = O(\alpha M - k) = O(y-y_0) \qquad &(3.4b)\\ U_1(\alpha, y) &= \frac{\alpha P_1}{\rho_0 c_0(k-\alpha M)} + \frac{iM'V_1}{M\alpha - k} = O(1) \qquad &(3.4c) \end{aligned}\right\} \text{ as } y \to y_0.$$

Let $\mathbf{Z}_2 \equiv \{P_2, V_2, U_2\}$ denote a solution which is linearly independent of $\mathbf{Z}_1$. If M'' is not identically zero in some entire neighbourhood† of y_0, the equations for P, U and V will each have a regular singular point at y_0 and P_2, V_2 and U_2 will be of the form

$$\left.\begin{aligned} P_2(\alpha, y) &= a + bP_1(\alpha, y)\ln(\alpha M - k) + O(\alpha M - k)\\ V_2(\alpha, y) &= a' + b'V_1(\alpha, y)\ln(\alpha M - k) + O(\alpha M - k)\\ U_2(\alpha, y) &= a'' + b''U_1(\alpha, y)\ln(\alpha M - k) + O(\alpha M - k) \end{aligned}\right\} \text{ as } y \to y_0, \tag{3.5}$$

where a, a', a'', b, b' and b'' are constants.

If the shear layer is symmetric about some plane, say $y = 0$ (i.e. if $M(-y) = M(y)$), the solutions (3.3) will reflect that symmetry and, as shown in appendix A, we can write

$$\mathbf{Z}_1(\alpha, y) = \{P_1(\alpha, |y|), (\operatorname{sgn} y) V_1(\alpha, |y|), U_1(\alpha, |y|)\} \quad \text{for} \quad M(-y) = M(y). \tag{3.6}$$

† If $M'' \equiv 0$ in some neighbourhood of y_0 only the equations for P and V will have regular singular points since y_0 will be an ordinary point of the equation for U. However, none of the three solutions P_2, V_2 and U_2 will then contain a logarithmic term.

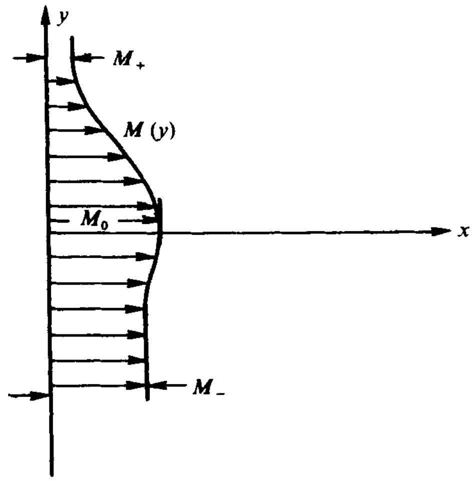

FIGURE 1. Parallel shear flow.

Apart from the two sets of classical solutions (3.3) and (3.5), (3.3*a–c*) possess a third, generalized† (weak) solution which (even when $M'' \equiv 0$) can be written as (Stakgold 1967)

$$\mathbf{Z}_1(\alpha, y) H(\alpha M - k) = \{P_1(\alpha, y) H(\alpha M - k), V_1(\alpha, y) H(\alpha M - k), U_1(\alpha, y) H(\alpha M - k)\}, \tag{3.7}$$

where H denotes the Heaviside function: $H(x) = 1$ for $x > 0$ and $H(x) = 0$ for $x < 0$. This can easily be verified by substituting the first component of (3.7) into the equation

$$P'' - \frac{2M'\alpha}{\alpha M - k} P' + [(\alpha M - k)^2 - \alpha^2] P = 0 \tag{3.8}$$

for P (which is obtained by eliminating U and V between (3.2*a–c*); see Betchov & Criminale 1967, p. 177), using (3.4*a*) together with the well-known relations for the delta function $\delta(x) = H'(x)$ [$f(x)\,\delta(x) = 0$ when $f(x) = O(x)$ as $x \to 0$] and $x\delta'(x) = -\delta(x)$ and then determining the corresponding solutions for V and U from (3.2). We could, of course, just as well start with the equation for V or the one for U.

When the mean flow is uniform the upwash velocity Fourier transform satisfies the equation

$$(k - \alpha M)\{V'' + [(k - \alpha M)^2 - \alpha^2]\, V\} = 0,$$

which is obtained by eliminating P and U between (3.2*a–c*) while taking care not to divide through by $k - \alpha M$. Since M is now constant and $x\delta(x) = 0$ this equation possesses the generalized solution $\delta(k - \alpha M) f(y)$, where f is an arbitrary function of y. Taking the inverse Fourier transform of this quantity, we find that it is precisely the solution that gives rise to the gust or vortical mode (2.4) for a uniform flow. We shall now show that (3.7) is the corresponding generalized solution that constitutes

† Such solutions satisfy the differential equations in the sense of a distribution rather than pointwise. However, they must be considered to obtain the complete set of solutions whenever the equations are the result of taking Fourier transforms. Solutions of this type were used in a related context by Case (1960). They are sometimes characterized as belonging to the discrete spectrum of the differential equations.

the primary component of a gust on a sheared mean flow. The change in character of the singular behaviour (from a delta function to a step function) results from the fact that the convected disturbances in a shear flow must consist of a range of wavenumbers for each frequency rather than a single wavenumber as they do when the flow is uniform.

As indicated in the introduction, we shall take the shear layer to be doubly infinite in the y direction and we shall suppose, in order to fix ideas, that the mean velocity becomes uniform (i.e. approaches a constant value) at large values of y and has a single maximum or minimum at some point within the layer, which we take without loss of generality to be $y = 0$ (see figure 1). Other cases of interest, which involve shear layers with monotonic velocity profiles or shear layers that are bounded on one or both sides by parallel walls, are actually easier to treat but will not be considered because they do not involve all of the features of the present example.

Apart from the singular solutions, equations (3.2) possess a set of solutions, say

$$\mathbf{Z}_o \equiv \{P_o(\alpha, y), V_o(\alpha, y), U_o(\alpha, y)\}, \tag{3.9}$$

that is a linear combination of the two linearly independent regular solutions $\mathbf{Z}_1$ and $\mathbf{Z}_2$ (since each component of $\mathbf{Z}$ satisfies a second-order differential equation which can have at most two linearly independent solutions) and behave either like outgoing waves or decay exponentially at large distances from the shear layer where the flow is uniform (i.e. they satisfy a radiation condition in this region; see Betchov & Criminale 1967, pp. 178–179). However, we shall in general have to allow these solutions to have a discontinuity at some point within the shear layer if they are to have outgoing-wave behaviour on both sides of this layer. We can always suppose that this discontinuity is at $y = 0$. Thus, when the mean flow is completely uniform, the outgoing-wave solutions can be taken as

$$\mathbf{Z}_o = \{\rho_0 c_0 A \exp i\tilde{\gamma}|y|, (A\tilde{\gamma} \operatorname{sgn} y \exp i\tilde{\gamma}|y|)/(k-\alpha M), (\alpha A \exp i\tilde{\gamma}|y|)/(k-\alpha M)\}, \tag{3.9a}$$

where the branch of the square root

$$\tilde{\gamma} \equiv [(k-\alpha M)^2 - \alpha^2]^{\frac{1}{2}} \tag{3.10}$$

is so chosen that its imaginary part is positive when its argument is negative and A is an arbitrary constant.

In the general case $\mathbf{Z}_o$ will always have the representation (3.9a) in the uniform flow region at large distances from the shear layer. The asymptotic representation of another solution, say $\mathbf{Z}_I$, that is linearly independent of $\mathbf{Z}_o$ can be obtained by replacing $\tilde{\gamma}$ by $-\tilde{\gamma}$; it will represent an incoming wave in the range of wavenumbers where $\tilde{\gamma}$ is real while otherwise it will be unbounded, i.e. infinite. Then when the shear layer is symmetric about $y = 0$ we can apply the same symmetry to arguments to $\mathbf{Z}_o$ and $\mathbf{Z}_I$ as were applied to $\mathbf{Z}_1$ and $\mathbf{Z}_2$ to develop the representation (3.6) and thereby show that $\mathbf{Z}_o$ can be represented as

$$\mathbf{Z}_o(\alpha, y) = \{P_o(\alpha, |y|), (\operatorname{sgn} y) V_o(\alpha, |y|), U_o(\alpha, |y|)\} \quad \text{for} \quad M(y) = M(-y). \tag{3.11}$$

3.2. *Construction of gust solution*

We shall now use the results of the previous section to construct the gust solution for a transversely sheared mean flow. To this end we notice that, since (3.7) and (3.9) satisfy the transformed equations (3.2), it follows from (3.1) and (3.3) that the linearized gasdynamic equations possess a solution, say

$$\bar{\zeta}_g(x,y)\,e^{-i\omega t} \equiv \{\bar{p}_g(x,y), \bar{v}_g(x,y), \bar{u}_g(x,y)\}\,e^{-i\omega t},$$

of the form

$$\bar{\zeta}_g(x,y) = \int_{k/M(y)}^{k/M_\pm} e^{i\alpha x} A^\pm(\alpha)\,\mathbf{Z}_1(\alpha,y)\,d\alpha + \int_{k/M_0 - i\epsilon}^{k/M_\pm - i\epsilon} e^{i\alpha x} B^\pm(\alpha)\,\mathbf{Z}_0(\alpha,y)\,d\alpha, \quad y \gtrless 0, \quad (3.12)$$

where the notation is meant to indicate that the plus (upper) signs apply when $y > 0$ and the minus signs apply when $y < 0$, $A^\pm$ and $B^\pm$ denote, as yet, arbitrary functions of α, ϵ is a small positive constant that can be set equal to zero after the contour integrals have been evaluated, M_0 is the Mach number at $y = 0$, and $M_\pm$ denote the limiting Mach numbers as $y \to \pm\infty$ (see figure 1).

This result is, in general, discontinuous at $y = 0$. In order to achieve continuity of the pressure and upwash velocity at this point we must take

$$\int_{k/M_0 - i\epsilon}^{k/M_+ - i\epsilon} e^{i\alpha x}[A^+(\alpha)\,P_1(\alpha, 0+) + B^+(\alpha)\,\;P_o(\alpha, 0+)]\,d\alpha$$
$$= \int_{k/M_0 - i\epsilon}^{k/M_- - i\epsilon} e^{i\alpha x}[A^-(\alpha)\,P_1(\alpha, 0-) + B^-(\alpha)\,P_o(\alpha, 0-)]\,d\alpha, \quad (3.13a)$$

$$\int_{k/M_0 - i\epsilon}^{k/M_+ - i\epsilon} e^{i\alpha x}[A^+(\alpha)\,V_1(\alpha, 0+) + B^+(\alpha)\,V_o(\alpha, 0+)]\,d\alpha$$
$$= \int_{k/M_0 - i\epsilon}^{k/M_- - i\epsilon} e^{i\alpha x}[A^-(\alpha)\,V_1(\alpha, 0-) + B^-(\alpha)\,V_o(\alpha, 0-)]\,d\alpha, \quad (3.13b)$$

where $0\pm$ denote the limits as $y \to 0$ from above/below. To determine $B^\pm$ (in terms of $A^\pm$) from these equations we must, in general, solve a coupled set of integral equations. But when the limiting velocities above and below the shear layer are equal (i.e. when $M_+ = M_-$) these equations are equivalent to the algebraic equations

$$\left.\begin{aligned} B^+(\alpha)\,P_o(\alpha, 0+) - B^-(\alpha)\,P_o(\alpha, 0-) &= -A^+(\alpha)\,P_1(\alpha, 0+) + A^-(\alpha)\,P_1(\alpha, 0-) \\ B^+(\alpha)\,V_o(\alpha, 0+) - B^-(\alpha)\,V_o(\alpha, 0-) &= -A^+(\alpha)\,V_1(\alpha, 0+) + A^-(\alpha)\,V_1(\alpha, 0-) \end{aligned}\right\} \quad (3.14)$$
$$\text{for} \quad M_+ = M_-,$$

which can be solved for $B^\pm$ to obtain

$$B^\pm(\alpha) = -A^\pm(\alpha)\,\Gamma^\pm_\mp(\alpha) + A^\mp(\alpha)\,\Gamma^\mp_\pm(\alpha), \quad (3.15)$$

where

$$\left.\begin{aligned} \Gamma^\pm_\mp(\alpha) &\equiv \frac{P_1(\alpha, 0\pm)\,V_o(\alpha, 0-) - V_1(\alpha, 0\pm)\,P_o(\alpha, 0-)}{P_o(\alpha, 0+)\,V_o(\alpha, 0-) - P_o(\alpha, 0-)\,V_o(\alpha, 0+)}, \\ \Gamma^\pm_\pm(\alpha) &= -\frac{P_1(\alpha, 0\mp)\,V_o(\alpha, 0+) - V_1(\alpha, 0\mp)\,P_o(\alpha, 0+)}{P_o(\alpha, 0+)\,V_o(\alpha, 0-) - P_o(\alpha, 0-)\,V_o(\alpha, 0+)}. \end{aligned}\right\} \quad (3.16)$$

When the velocity profile is symmetric about $y = 0$ it follows from (3.6) and (3.11) that

$$\Gamma^\pm_\mp(\alpha) = \Gamma^\pm_\pm(\alpha) \equiv \Gamma^\pm(\alpha) = \frac{1}{2}\left[\frac{P_1(\alpha, 0)}{P_o(\alpha, 0)} \pm \frac{V_1(\alpha, 0+)}{V_o(\alpha, 0+)}\right] \quad \text{for} \quad M(y) = M(-y). \quad (3.17)$$

Since $M'(0) = 0$ by construction, (3.2*b*) implies that

$$\alpha P(\alpha, 0\pm) = \rho_0 c_0 (k - \alpha M_0)\,U(\alpha, 0\pm), \quad (3.18)$$

which, in turn, implies that

$$P_1(\alpha,0+):P_1(\alpha,0-):P_o(\alpha,0+):P_o(\alpha,0-)$$
$$= U_1(\alpha,0+):U_1(\alpha,0-):U_o(a,0+):U_o(\alpha,0-).$$

Hence it follows from (3.12) and (3.14) that the relations (3.15) will also ensure that the axial velocity $\bar{u}_g$ will be continuous at $y = 0$ when $M_+ = M_-$. When $M_+ \neq M_-$ the solutions to (3.13) may have enough arbitrariness that we can impose an axial velocity continuity requirement but I have not been able to prove this.

Notice that (3.12) involves only wavenumber components that lie between $k/M_{\min}$ and $k/M_{\max}$, where $M_{\max}$ and $M_{\min}$ are the maximum and minimum Mach numbers of the flow. *It therefore contains only waves travelling in the axial direction with phase speeds that lie between the maximum and minimum flow velocity,* so that the waves comprising the second term in (3.12) will vanish at infinity when the mean flow is everywhere subsonic, or more precisely, when the maximum change in the mean flow Mach number across the shear layer is less than one. This can be seen by recalling that these waves behave like (3.9*a*) at infinity and noting that the square root (3.10) is a strictly positive imaginary quantity for the range of wavenumbers being considered.

The most general solution to the linearized gasdynamic equations that consists entirely of wavenumbers in the range $k/M_{\max} < \alpha < k/M_{\min}$ can be obtained by adding to (3.12) an arbitrary linear combination of the incoming-wave solutions $\mathbf{Z}_I$ that possess wavenumbers in this range. But since the first term in (3.12) goes to zero as $y \rightarrow \pm\infty$ while, as can be seen from (3.10) and the remarks in the paragraph immediately following this equation, $\mathbf{Z}_I$ will become infinite as $y \rightarrow \pm\infty$ whenever $M_{\max} - M_{\min} < 1$ (which is the case of primary interest here), the latter solution (i.e. (3.12) plus the linear combination of $\mathbf{Z}_I$) will also become infinite at infinity and therefore not satisfy the boundedness requirement that we intend to impose on the gust solution. The supersonic mean flow case is more subtle and will not be pursued here. We merely note in passing that the gust will then contain components that do not decay at infinity and therefore give rise to the Mach wave radiation first discussed by Phillips (1960). The role played by the incident waves will be discussed further in § 3.3.

Thus, assuming that $M_{\max} - M_{\min}$, the relative Mach number change across the shear layer, is less than one, (3.12) with the $B^{\pm}$ determined by (3.13) or by (3.15) when $M_+ = M_-$ *represents the most general continuous*† *bounded solution consisting entirely of waves with axial phase velocities lying between the maximum and minimum mean flow velocity.* It is clear that any convected disturbance (i.e. any disturbance that moves downstream with the local mean flow velocity) must comprise all the wavenumbers in this range.

The vorticity amplitude

$$\bar{\omega}_g \equiv \partial\bar{v}_g/\partial x - \partial\bar{u}_g/\partial y \tag{3.19}$$

associated with the gust solution (3.12) is given by

$$\bar{\omega}_g = (k/M)'U_1(k/M(y),y)\exp[ikx/M(y)]\,A^{\pm}(k/M(y)) + \int_{k/M(y)}^{k/M_{\pm}} e^{i\alpha x}A^{\pm}(\alpha)\,\Pi_1(\alpha,y)\,d\alpha$$
$$+\int_{k/M_0-i\epsilon}^{k/M_{\pm}-i\epsilon} e^{i\alpha x}B^{\pm}(\alpha)\,\Pi_o(\alpha,y)\,d\alpha, \quad y \gtrless 0, \tag{3.20}$$

† With the possible exception of the tangential velocity at $y = 0$ when $M_+ \neq M_-$.

where

$$\Pi(\alpha, y) \equiv i\alpha V(\alpha, y) - U'(\alpha, y). \tag{3.21}$$

The differentiation of the logarithmic term in U_o (recall that this solution is a linear combination of U_1 and U_2 and that U_2 has a logarithmic singularity at y_0) will cause Π_o to have a pole at the point y_0 where $\alpha = k/M$. But, since the integration contour in the second integral in (3.20), which lies below the real axis, must be closed in the lower half-plane when x is negative, this pole will not contribute to the asymptotic value of the integral as $x \to -\infty$. Hence it follows from the theory of Fourier transforms that as long as $A^{\pm}(\alpha)$ are bounded

$$\zeta_g = O(\ln|x|/x) \quad \text{as} \quad x \to -\infty \tag{3.22}$$

and

$$\overline{\omega}_g = \Omega(y) \exp[ikx/M(y)] + O(\ln|x|/x) \quad \text{as} \quad x \to -\infty,$$

where we have put

$$\frac{\Omega(y)}{[k/M(y)]' U_1(k/M(y), y)} \equiv A^{\pm}(k/M(y)), \quad y \gtrless 0. \tag{3.23}$$

Thus as indicated in the introduction, the vorticity ω_g associated with the gust solution (3.12) behaves like the frozen convected disturbance

$$\Omega(y) \exp\{-ik[c_0 t - x/M(y)]\} \tag{3.24}$$

at large distances upstream. It is clear from the development that (3.12) has the smallest range of wavenumbers of any continuous bounded solution that exhibits this behaviour. It therefore corresponds to the definition of the gust solution given near the end of § 2. In a uniform flow the vorticity is of the form (3.24) at all points of the flow. But when the flow is non-uniform this behaviour will occur only far upstream.

The transverse vorticity distribution $\Omega(y)$ can be specified more or less arbitrarily when the flow is uniform and, since we can always select $A^{\pm}$ in accordance with (3.23) for any Ω, we now see that this is also the case when the flow is non-uniform. Thus let $f(y) \equiv k/M(y)$. Then $f(y)$ has a unique† inverse $\eta^+ \equiv f^{-1}$ in the range $y > 0$ and a unique inverse $\eta^- \equiv f^{-1}$ in the range $y < 0$. Consequently

$$y = \eta^{\pm}(k/M(y)), \quad y \gtrless 0, \tag{3.25}$$

and (3.23) will be satisfied for any Ω if we define $A^{\pm}(\alpha)$ by

$$A^{\pm}(\alpha) = \frac{d\eta^{\pm}(\alpha)}{d\alpha} \frac{\Omega(\eta^{\pm}(\alpha))}{U_1(\alpha, \eta^{\pm}(\alpha))}. \tag{3.26}$$

Since this determines $A^{\pm}$ uniquely, it follows from (3.12) and (3.15) that the gust solution is itself uniquely determined by specifying the upstream transverse vorticity distribution $\Omega(y)$.

3.3. *Discussion and final results*

The solution to any given problem will be the sum of the gust solution (3.12), which is determined by $\Omega(y)$, and a 'non-gust' component which is determined by conditions imposed at the boundaries of the problem. Although it might at first appear that

† Since the inverse function is multi-valued on the full range $-\infty < y < \infty$ we must consider separately each of the ranges where it is single valued if we are to write down unambiguous formulae.

$\Omega(y)$ represents the upstream distribution [via (3.19), (3.23) and (3.24)] of *only* the gust component of the vorticity field, it is clear that in any problem involving the scattering of vortical disturbances, as opposed to incident acoustic waves, the non-gust component of the solution will consist only of evanescent waves or outgoing cylindrical waves that will also decay at infinity. Consequently, only the gust component of the solution will contribute to the upstream vorticity field and $\Omega(y)$ will then represent the transverse distribution of the *total* upstream vorticity.

On the other hand, an imposed upstream acoustic field must be composed of the incoming-wave solutions $\mathbf{Z}_I$ whose axial wavenumbers lie in the range where these solutions have wavelike behaviour at infinity, i.e. in the range where the square root (3.10) is real [see remarks in paragraph following (3.10)]. As we have already indicated, this range will not include the wavenumbers that contribute to the gust solution (3.12) as long as we restrict our attention to the case where the change in the Mach number across the shear layer is less than one. Moreover, it is not hard to show that the solutions whose wavenumbers lie outside the wavenumber range in (3.12) will then have zero vorticity far upstream, so that $\Omega(y)$ will again represent the transverse distribution of the total upstream vorticity.

The unbounded $\mathbf{Z}_I$ solutions (i.e. the solutions that correspond to the wavenumber range where the square root (3.10) becomes imaginary) are usually of little interest in scattering problems on flows that extend to infinity. It should therefore always be possible to distinguish the acoustical and vortical solutions in the distant upstream region of a subsonic mean flow.

In order to obtain explicit formulae for the gust solution we now restrict our attention to the case where $M^+ = M^- \equiv M_\infty$. Then substituting (3.15) and (3.26) into (3.12) yields

$$\begin{aligned}\boldsymbol{\zeta}_g(x,y) = &\int_{k/M(y)}^{k/M_\infty} e^{i\alpha x}\frac{\Omega(\eta^+(\alpha))}{U_1(\alpha,\eta^+(\alpha))}[\mathbf{Z}_1(\alpha,y)-\Gamma_+^+(\alpha)\,\mathbf{Z}_o(\alpha,y)]\frac{d\eta^+(\alpha)}{d\alpha}\,d\alpha\\ &-\int_{k/M_0}^{k/M(y)} e^{i\alpha x}\frac{\Omega(\eta^+(\alpha))}{U_1(\alpha,\eta^+(\alpha))}\Gamma_+^+(\alpha)\,\mathbf{Z}_o(\alpha,y)\frac{d\eta^+(\alpha)}{d\alpha}\,d\alpha\\ &+\int_{k/M_0}^{k/M_\infty} e^{i\alpha x}\frac{\Omega(\eta^-(\alpha))}{U_1(\alpha,\eta^-(\alpha))}\Gamma_+^-(\alpha)\,\mathbf{Z}_o(\alpha,y)\frac{d\eta^-(\alpha)}{d\alpha}\,d\alpha \quad \text{for} \quad y>0.\end{aligned}$$

This can be put into a more concise and revealing form by taking $\eta^\pm(\alpha)$ as the integration variable (which is possible since each of these quantities is single valued) to obtain

$$\begin{aligned}\boldsymbol{\zeta}_g(x,y) = &\int_y^\infty \exp[ikx/M(\eta)]\,\Omega(\eta)\frac{\mathbf{Z}_1(k/M(\eta),y)-\gamma(\eta)\,\mathbf{Z}_o(k/M(\eta),y)}{U_1(k/M(\eta),\eta)}\,d\eta\\ &-\int_{-\infty}^{y} \exp[ikx/M(\eta)]\,\Omega(\eta)\,\gamma(\eta)\frac{\mathbf{Z}_o(k/M(\eta),y)}{U_1(k/M(\eta),\eta)}\,d\eta \quad \text{for} \quad y>0, \qquad (3.27)\end{aligned}$$

where we have put

$$\gamma(\eta) \equiv \Gamma_+^\pm(k/M(\eta)) \quad \text{for} \quad \eta \gtrless 0 \qquad (3.28)$$

and dropped the superscripts on η since the positive and negative ranges are now determined by the limits of integration. When the mean flow is symmetric it follows from (3.17) and (3.18) that

$$\gamma(\eta) = \frac{1}{2}\left[\frac{P_1(k/M(|\eta|), 0+)}{P_o(k/M(|\eta|), 0)} + \operatorname{sgn}\eta \frac{V_1(k/M(|\eta|), 0+)}{V_o(k/M(|\eta|), 0+)}\right]$$

$$= \frac{1}{2}\left[\frac{U_1(k/M(|\eta|), 0+)}{U_o(k/M(|\eta|), 0+)} + \operatorname{sgn}\eta \frac{V_1(k/M(|\eta|), 0+)}{V_o(k/M(|\eta|), 0+)}\right] \quad \text{for} \quad M(\eta) = M(-\eta), \tag{3.29}$$

and in order to be consistent we must choose the normalization for U_1 in (3.27) such that $U_1(k/M(\eta), \eta) = U_1(k/M(\eta), -\eta)$ when $\eta < 0$.

For scattering problems we are primarily interested in the transverse velocity component of (3.27):

$$\bar{v}_g = \int_y^\infty \exp[ikx/M(\eta)]\,\Omega(\eta)\left[\frac{V_1(k/M(\eta), y) - \gamma(\eta)\,V_o(k/M(\eta), y)}{U_1(k/M(\eta), \eta)}\right] d\eta$$

$$- \int_{-\infty}^{y} \exp[ikx/M(\eta)]\,\Omega(\eta)\,\gamma(\eta)\frac{V_o(k/M(\eta), y)}{U_1(k/M(\eta), \eta)}\,d\eta \quad \text{for} \quad y > 0. \tag{3.30}$$

Similar manipulation of (3.2) yields

$$\bar{\omega}_g = \Omega(y)\exp[ikx/M(y)]$$

$$+ \int_y^\infty \exp[ikx/M(\eta)]\,\Omega(\eta)\left[\frac{\Pi_1(k/M(\eta), y) - \gamma(\eta)\,\Pi_o(k/M(\eta), y)}{U_1(k/M(\eta), \eta)}\right] d\eta$$

$$- \int_{-\infty}^{y} \exp[ikx/M(\eta)]\,\Omega(\eta)\,\gamma(\eta)\,\frac{\Pi_o(k/M(\eta), y)}{U_1(k/M(\eta), \eta)}\,d\eta \quad \text{for} \quad y > 0, \tag{3.31}$$

where Π is defined by (3.21).

The results of this section are best understood when they are compared with the corresponding uniform mean flow formulae of § 2. These results show that, as in the case considered in § 2, the velocity field is induced by a transverse distribution

$$\Omega(\eta)\exp[i(kx/M(\eta) - \omega t)]$$

of frozen disturbances that are convected downstream at the *local* mean flow velocity and are related to the upstream vorticity distribution. On the other hand, these disturbances do not, as in the constant mean flow case, themselves coincide with the actual vorticity field but, as we shall show subsequently, are rather related to a certain quantity that is composed of the particle displacement as well as the vorticity. The vorticity field is now the sum of these convected disturbances and a term that is merely induced by these disturbances in the same fashion as the velocity field. Moreover, the pressure field, rather than being identically zero, is now also induced by these convected disturbances.

4. Comparison of transversely sheared mean flow solutions with uniform flow results

We have shown that (3.20) and (3.31) bear a considerable resemblance to the uniform flow equations (2.4) and (2.5). In fact, in the limit as $M(y)$ approaches a constant while x remains finite† we can factor $e^{ikx/M}$ out of the integrals and put (3.27) and (3.31) into the form

$$\bar{\boldsymbol{\zeta}}_g = \{\bar{p}_g, \bar{v}_g, \bar{u}_g\} = \tilde{\mathbf{Z}}_e(y)\, e^{ikx/M}, \tag{4.1}$$

$$\bar{\omega}_g = \Omega_e(y)\, e^{ikx/M}, \tag{4.2}$$

where

$$\begin{aligned}\tilde{\mathbf{Z}}_e &= \{\tilde{P}_e(y), \tilde{V}_e(y), \tilde{U}_e(y)\}\\ &\equiv \int_y^\infty \Omega(\eta)\left[\frac{\mathbf{Z}_1(k/M(\eta), y) - \gamma(\eta)\,\mathbf{Z}_o(k/M(\eta), y)}{U_1(k/M(\eta), \eta)}\right] d\eta\\ &\quad - \int_{-\infty}^y \Omega(\eta)\,\gamma(\eta)\,\frac{\mathbf{Z}_o(k/M(\eta), y)}{U_1(k/M(\eta), \eta)}\, d\eta \quad \text{for} \quad y > 0\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}\Omega_e(y) &= \Omega(y) + \int_y^\infty \Omega(\eta)\left[\frac{\Pi_1(k/M(\eta), y) - \gamma(\eta)\,\Pi_o(k/M(\eta), y)}{U_1(k/M(\eta), \eta)}\right] d\eta\\ &\quad - \int_{-\infty}^y \Omega(\eta)\,\gamma(\eta)\,\frac{\Pi_o(k/M(\eta), y)}{U_1(k/M(\eta), \eta)}\, d\eta \quad \text{for} \quad y > 0,\end{aligned} \tag{4.4}$$

with $\mathbf{Z}_1$, $\mathbf{Z}_o$ and Π given by (3.6), (3.11) and (3.21), respectively. Then eliminating V between (3.2*a*, *b*) shows that

$$\frac{P(\alpha, y)}{\rho_0 c_0\, U(\alpha, y)} = \left[\frac{k}{\alpha} - M(y)\right] \Big/ \left[1 - \frac{M'}{\alpha(k - \alpha M)}\frac{P'}{P}\right].$$

But since the denominator on the right side of this expression will not, in general, vanish as $M \to k/\alpha$

$$\frac{P(k/M(\eta), y)}{U(k/M(\eta), y)} = O(M(\eta) - M(y)) \quad \text{as} \quad M \to \text{a constant.}$$

Consequently, it follows from (4.3) that $\tilde{P}_e$ will be negligible compared with $\tilde{U}_e$ as M becomes constant, so that $p_g \approx 0$ and (2.3) holds.

It follows from (3.2) that

$$\frac{V'(k/M(\eta), y)}{U(k/M(\eta), y)} = \frac{k}{iM}\frac{U(k/M(\eta), y)}{U(k/M(\eta), \eta)} + O\left([M(\eta) - M(y)]\,\frac{U(k/M(\eta), y)}{U(k/M(\eta), \eta)}\right).$$

Then since (3.4*b*) implies that

$$V_1(k/M(y), y) = 0, \tag{4.5}$$

differentiating the second component of (4.3) (i.e. $\tilde{V}_e$) with respect to y and comparing with the third we find that $\tilde{U}_e$ and $\tilde{V}_e$ are related to one another by (2.6). It can also be shown that this relation holds when $y < 0$.

Eliminating P between (3.2*b*, *c*) yields

$$U(\alpha, y) = i\,\frac{(k - \alpha M)\, M' V + \alpha V'}{\alpha^2 - (k - \alpha M)^2}. \tag{4.6}$$

Then since M' will be small and $M(y)$ will approach $M(\eta)$ when M approaches a constant value, this implies that

$$U(k/M(\eta), y) \approx (iM/k)\, V'(k/M(\eta), y). \tag{4.7}$$

† If we allowed x to become infinite, the product of x with the variable part of $1/M(\eta)$ need never be small and we could not treat $\exp\,[ikx/M(\eta)]$ as independent of η.

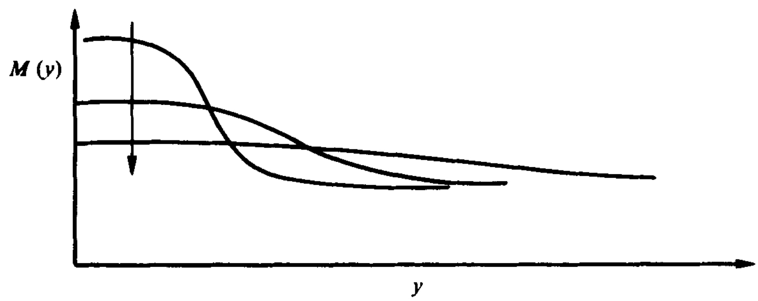

FIGURE 2. Approach of Mach number profile to a constant value needed to ensure uniformity of limit.

Consequently, it follows from (3.21) and (4.5) that we can interchange the order of integration and differentiation in (4.4) to obtain

$$\Omega_e(y) = \frac{iM}{k}\left[\left(\frac{k}{M}\right)^2 \tilde{V}_e(y) - \frac{d^2}{dy^2}\tilde{V}_e(y)\right],$$

where $\tilde{V}_e$ is the second component of (4.3) with $\mathbf{Z}_1$ and $\mathbf{Z}_0$ given by (3.6) and (3.11), respectively. It can also be shown that this equation holds when $y < 0$. It is a second-order ordinary differential equation for $\tilde{V}_e$ that can easily be integrated. Its general solution will involve an arbitrary linear combination of the two homogeneous solutions $e^{\pm ky/M}$ (and is, in fact, given by the equation immediately following (2.6) with Ω replaced by Ω_e). However, as indicated in § 2, the only solution for which $\tilde{V}_e$ and $\tilde{U}_e$, which, as we have just shown, is now related to $\tilde{V}_e$ by (2.6), will be everywhere continuous and bounded is given by (2.7) (with Ω, of course, replaced by Ω_e).

We have now established that, in the limit as M becomes constant, the pressure, velocity and vorticity do indeed satisfy the relations (2.3)–(2.7) that determine the unsteady vortical modes on a uniform flow. Thus as M becomes constant the pressure fluctuations of the gust vanish while its velocity and vorticity become purely convected disturbances whose amplitudes $\tilde{U}_e$, $\tilde{V}_e$ and Ω_e are connected by (2.6) and (2.7). Equation (2.6), of course, implies that the velocity is solenoidal.

On the other hand, (4.4) shows that the vorticity amplitude Ω_e that appears in these relations will not be the same as the specified upstream vorticity amplitude $\Omega(y)$ unless $\Pi \to 0$ as M becomes constant, which, as we shall show, will usually not occur. This non-uniform behaviour is related to the discontinuous change in the asymptotic behaviour of the velocity fluctuations. Thus (3.22) shows that $\tilde{\zeta}_g$ will always decay as $x \to -\infty$ if M is non-constant while (4.1) shows that $\tilde{\zeta}_g$ will never decay as $x \to -\infty$ when M is constant. Consequently the order of taking the limits $x \to -\infty$ and $M \to$ constant cannot be interchanged. In view of this behaviour it is not surprising that

$$\lim_{x\to-\infty} \lim_{M\to\text{const.}} \bar{\omega}_g \neq \lim_{M\to\text{const.}} \lim_{x\to-\infty} \bar{\omega}_g.$$

This non-uniform limit causes a certain amount of ambiguity concerning the correct method of specifying the amplitude of the incident gust on a constant mean flow. After all, no real flow is ever perfectly constant and the above results indicate that an incident gust may be easily distorted by even a very small non-uniformity. It therefore seems worthwhile to determine when we can expect Ω_e to approach $\Omega(y)$.

This is done in appendix B, where it is shown that Ω_e will approach Ω as M becomes constant if we require that the radius of curvature of the mean velocity profile simultaneously becomes large relative to a wavelength in the sense that

$$M''(\eta)\,M(\eta)/[M'(\eta)\,k] \to 0$$

as M becomes constant. The approach to the constant mean velocity limit must therefore occur in the manner indicated in figure 2. It is also shown in appendix B that (3.30) will reduce directly to (2.4a) with $\tilde{V}$ given by (2.7) when the limit is approached in this fashion.

5. Further interpretation of results

As we have already indicated, there are no purely convected physical disturbances in a transversely sheared mean flow as there are when the flow is uniform (unless, as shown by Möhring (1976), the mean shear is constant). However, there is a quantity related to the vorticity which is convected by the mean flow. Thus, for the small amplitude, constant mean density two-dimensional motion being considered, the linearized vorticity equation becomes

$$\frac{D_0}{Dt}\left[\omega - \frac{p}{\rho_0 c_0^2}\,\omega_0\right] + v\frac{d\omega_0}{dy} = 0, \tag{5.1}$$

where

$$\frac{D_0}{Dt} \equiv \frac{\partial}{\partial t} + c_0 M(y)\frac{\partial}{\partial x}$$

and $\omega_0 \equiv -c_0 M'$ is the mean vorticity. Upon introducing the particle displacement ξ in the usual way by

$$v = D_0\xi/Dt, \tag{5.2}$$

this becomes

$$D_0\omega^*/Dt = 0,$$

where

$$\omega^* \equiv \omega + M'p/\rho_0 c_0 - M'' c_0 \xi. \tag{5.3}$$

Consequently ω^* is simply convected by the mean flow.

We again restrict the discussion to a single harmonic component of the unsteady motion. Then the axial Fourier transform of ω^* (i.e. the Fourier transform with respect to x) is

$$\Pi(\alpha, y) + M'\frac{P(\alpha, y)}{\rho_0 c_0} - \frac{iM'' V(\alpha, y)}{(k-\alpha M)}, \tag{5.4}$$

where Π is defined by (3.21). But (3.2) shows that $k-\alpha M$ times this quantity will equal zero and therefore that (5.4) will equal zero at all points where $k-\alpha M \neq 0$. Consequently, no regular (i.e. non-distributional) solution to the linearized gas-dynamic equations will contribute to ω^*. (But since $(k-\alpha M)\,\delta(k-\alpha M) = 0$ this does not rule out the possibility of the distributional solution contributing to ω^*.)

Now the solution, say $\boldsymbol{\zeta} e^{-i\omega t} = \{\bar{p}e^{-i\omega t},\ \bar{v}e^{-i\omega t},\ \bar{u}e^{-i\omega t}\}$, to any boundary-value problem will, in general, consist of a regular solution plus the distributional solution $\bar{\boldsymbol{\zeta}}_g e^{-i\omega t}$ defined in (3.12). Then

$$\begin{aligned}\bar{\omega}^* &= \bar{\omega} + M'\bar{p}/\rho_0 c_0 - M'' c_0 \bar{\xi} \\ &= \bar{\omega}_g + M'\bar{p}_g/\rho_0 c_0 - M'' c_0 \bar{\xi}_g \equiv \bar{\omega}_g^*,\end{aligned} \tag{5.5}$$

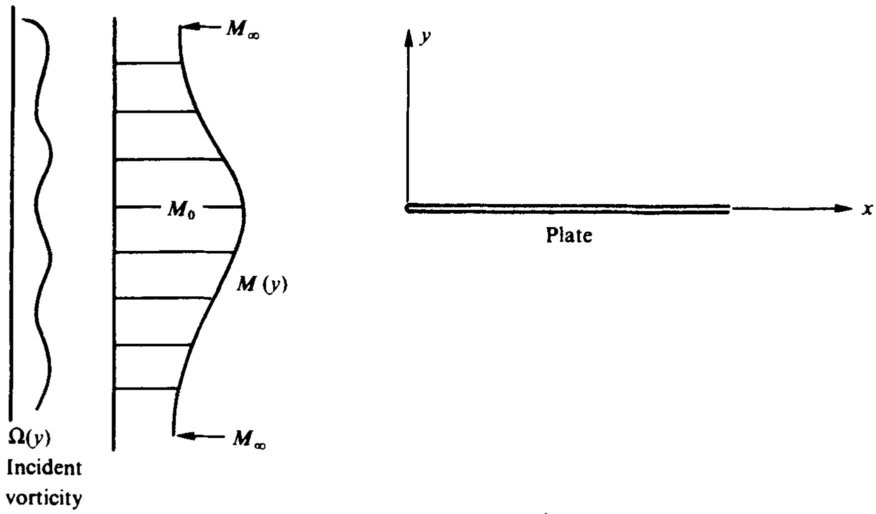

FIGURE 3. Geometry for scattering problem.

where

$$\left.\begin{aligned}\bar{\omega} &\equiv \partial\bar{v}/\partial x - \partial\bar{u}/\partial y,\\ \bar{v} &= c_0(-ik + M\,\partial/\partial x)\,\bar{\xi}\\ \bar{v}_g &= c_0(-ik + M\,\partial/\partial x)\,\bar{\xi}_g.\end{aligned}\right\} \tag{5.6}$$

and

On the other hand ξ is defined by (5.2) only to within an arbitrary function of the form $f(y)\exp[i(kx/M - \omega t)]$. Its definition can be made unique by using the inverse

$$\bar{\xi} = \frac{i}{c_0}\int_{-\infty-i\epsilon}^{\infty-i\epsilon} e^{i\alpha x}\,\frac{V(\alpha, y)}{(k-\alpha M)}\,d\alpha, \quad 0 < \epsilon \ll 1,$$

of the operator on the right side of (5.6) to define $\bar{\xi}$. Then on inserting (3.31) and the appropriate components of (3.27) into (5.5), we find that

$$\bar{\omega}^* = \Omega(y)\exp[ikx/M(y)].$$

Consequently $\bar{\omega}^*$ is just the convected portion of the gust vorticity amplitude $\bar{\omega}_g$. More important, ω^* is precisely the convected disturbance that was shown in § 3.3 to induce the entire unsteady flow field of the gust! It is also worth noting [see remarks following (3.26)] that $\bar{\omega} \to \bar{\omega}_g \to \bar{\omega}^*$ as $x \to -\infty$.

6. Scattering of a gust by a half-plane

6.1. *Construction of solution*

In order to illustrate the ideas of the previous sections we shall consider the problem of a gust incident on a semi-infinite plate in a subsonic sheared mean flow as shown in figure 3. We suppose that the plate is located at $y = 0$ (with its leading edge at $x = 0$) and that the mean flow is symmetric about $y = 0$. The unsteady motion is assumed to result from an upstream convected vorticity distribution $\Omega(y)\,e^{ikx/M(y)}$, which we shall suppose also to be symmetric with respect to $y = 0$. The solution $\bar{\boldsymbol{\zeta}} = \{\bar{p}, \bar{v}, \bar{u}\}$ to the problem is equal to the sum of the gust solution (3.27) and a scattered part

$\bar{\zeta}_a \equiv \{\bar{p}_a, \bar{v}_a, \bar{u}_a\}$ which has outgoing-wave behaviour at infinity (i.e. it satisfies a radiation condition) and has an upwash component at the plate that is equal and opposite to $\bar{v}_g$. In view of the symmetry of the problem we can write this solution in the form

$$\bar{p}_a(x,y) = \operatorname{sgn} y \int_{-\infty}^{\infty} e^{i\alpha x} A(\alpha)\, P_o(\alpha, |y|)\, d\alpha, \tag{6.1a}$$

$$\bar{v}_a = \int_{-\infty}^{\infty} e^{i\alpha x} A(\alpha)\, V_o(\alpha, |y|)\, d\alpha, \tag{6.1b}$$

where as before the subscript o is used to denote the solution with outgoing wave behaviour at infinity. The boundary conditions on $y = 0$ are $\bar{p}_a = 0$ for $x < 0$ and $\bar{v}_a = -\bar{v}_g$ for $x > 0$, where $\bar{v}_g$ is given by (3.30). Consequently, the problem amounts to solving the dual integral equations

$$\int_{-\infty}^{\infty} e^{i\alpha x} A(\alpha)\, P_o(\alpha, 0+)\, d\alpha = 0, \quad x < 0,$$

$$\int_{-\infty}^{\infty} e^{i\alpha x} A(\alpha)\, V_o(\alpha, 0+)\, d\alpha = -\bar{v}_g, \quad x > 0.$$

This can be accomplished by using the Wiener–Hopf technique to obtain (see Noble 1958, pp. 220ff.)

$$A(\alpha) = G_-(\alpha)/P_o(\alpha, 0+)\, \kappa_-(\alpha), \tag{6.2}$$

where $G_\pm(\alpha)$ denote analytic functions in the upper/lower half α plane that vanish at infinity and $\kappa_\pm(\alpha)$ denote non-zero analytic functions in the upper/lower half-plane with algebraic behaviour at infinity. They are uniquely determined (to within an irrelevant multiplicative constant) by their behaviour along the real axis, which is given by

$$\kappa_-(\alpha)/\kappa_+(\alpha) = V_o(\alpha, 0+)/P_o(\alpha, 0+) = \frac{1}{i\rho_0 c_0 (k - \alpha M_0)} \frac{P_o'(\alpha, 0)}{P_o(\alpha, 0)}, \qquad \operatorname{Im}\alpha = 0, \quad -\infty < \operatorname{Re}\alpha < \infty, \tag{6.3}$$

and

$$G_-(\alpha) - G_+(\alpha) = \kappa_+(\alpha)\, F_-(\alpha), \quad \operatorname{Im}\alpha = 0, \quad -\infty < \operatorname{Re}\alpha < \infty, \tag{6.4}$$

where

$$F_-(\alpha) = -\frac{1}{2\pi} \int_0^{\infty} e^{-i\alpha x}\, \bar{v}_g(x, 0)\, dx, \tag{6.5}$$

where we have used (3.2a) to obtain the right-hand part of (6.3) and where as usual we suppose that k has a small positive imaginary part that will be put equal to zero at the end of the analysis.

When Ω is symmetric (3.29) and (3.30) imply

$$\begin{aligned} \bar{v}_g(x, 0) &= \int_0^{\infty} \exp\left[ikx/M(\eta)\right] \frac{\Omega(\eta)}{U_1(k/M(\eta), \eta)} \left[V_1(k/M(\eta), 0+) - \frac{P_1(k/M(\eta), 0+)}{P_o(k/M(\eta), 0+)} V_o(k/M(\eta), 0+)\right] d\eta \\ &= \int_{k/M_0}^{k/M_\infty} e^{i\alpha x}\, \Omega(\eta(\alpha))\, R(\alpha) \frac{d\eta(\alpha)}{d\alpha}\, d\alpha, \end{aligned} \tag{6.6}$$

where we have put

$$R(\alpha) \equiv \frac{V_1(\alpha, 0+)}{U_1(\alpha, \eta(\alpha))} - \frac{P_1(\alpha, 0+)}{P_o(\alpha, 0+)} \frac{V_o(\alpha, 0+)}{U_1(\alpha, \eta(\alpha))}$$

and reinserted the original variable of integration. Inserting (6.6) into (6.5) and carrying out the integration with respect to x yields

$$F_-(\alpha) = \lim_{\delta \to 0+} \frac{1}{2\pi i} \int_{k/M_0}^{k/M_\infty} \frac{\Omega(\eta(\alpha'))}{\alpha' - \alpha + i\delta} R(\alpha') \frac{d\eta(\alpha')}{d\alpha'} d\alpha'.$$

Hence it follows from the Plemelj formulae (Woods 1961) that

$$F_-(\alpha) = F_+(\alpha) - [H(\alpha - k/M_0) - H(\alpha - k/M_\infty)]\, \Omega(\eta(\alpha))\, R(\alpha)\, d\eta(\alpha)/d\alpha,$$

where H still denotes the Heaviside function and F_+ is analytic and bounded at infinity in the upper half α plane. Inserting this into (6.4) yields

$$G_-(\alpha) - K_+(\alpha) = -[H(\alpha - k/M_0) - H(\alpha - k/M_\infty)]\, \kappa_+(\alpha)\, \Omega(\eta(\alpha))\, R(\alpha)\, d\eta(\alpha)/d\alpha,$$

where $K_+(\alpha) \equiv F_+(\alpha)\,\kappa_+(\alpha) + G_+(\alpha)$ is an analytic function in the upper half α plane with algebraic behaviour at infinity. Applying the Plemelj formulae (Woods 1961) to this result therefore yields

$$G_-(\alpha) = \lim_{\delta \to 0+} \frac{1}{2\pi i} \int_{k/M_0}^{k/M_\infty} \frac{\kappa_+(\tilde{\alpha})\, \Omega(\eta(\tilde{\alpha}))}{\tilde{\alpha} - \alpha + i\delta} R(\tilde{\alpha}) \frac{d\eta(\tilde{\alpha})}{d\tilde{\alpha}} d\tilde{\alpha}$$

$$= \frac{1}{2\pi i} \int_0^\infty M(\eta) \frac{\kappa_+(k/M(\eta))\, \Omega(\eta)}{k - \alpha M(\eta)} Q(\eta)\, d\eta,$$

where we have put

$$Q(\eta) \equiv R(k/M(\eta)) = \frac{V_1(k/M(\eta), 0)}{U_1(k/M(\eta), \eta)} - \frac{P_1(k/M(\eta), 0+)}{P_o(k/M(\eta), 0+)} \frac{V_o(k/M(\eta), 0+)}{U_1(k/M(\eta), \eta)} \tag{6.7}$$

and after returning to η as the variable of integration we have dropped the notation $\lim_{\delta \to 0+}$ since this limit is already accounted for by the fact that k is assumed to have a small positive imaginary part that will be put equal to zero at the end of the analysis.

Inserting this together with (6.2) into (6.1) and using (6.3) shows that $\bar{p}_a(x, y|\eta)$, the pressure fluctuation at the point (x, y) due to the incident vorticity at the height η above the plate, is given by

$$\bar{p}_a(x, y|\eta) = \operatorname{sgn} y \frac{\Omega(\eta)\, M(\eta)\, Q(\eta)}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{i\alpha x} \kappa_+(k/M(\eta))}{[k - \alpha M(\eta)]\, \kappa_+(\alpha)} \frac{P_o(\alpha, |y|)}{V_o(\alpha, 0+)} d\alpha. \tag{6.8}$$

Naturally

$$\bar{p}_a(x, y) = \int_0^\infty \bar{p}_a(x, y|\eta)\, d\eta. \tag{6.9}$$

6.2. *Acoustic radiation*

At large distances from the plate (i.e. for (x, y) in the radiation field) $\bar{p} \to \bar{p}_a$,

$$P_o(\alpha, |y|) \sim \rho_0 c_0 C(\alpha) \exp\{i[(k - \alpha M_\infty)^2 - \alpha^2]^{\frac{1}{2}} |y|\} \tag{6.10}$$

and the method of stationary phase (Carrier, Krook & Pearson 1966, p. 274) can be used to evaluate the integral in (6.8). For simplicity we suppose that the medium is at rest at infinity (i.e. that $M_\infty = 0$). Then (6.8) becomes

$$\frac{\bar{p}(x,y|\eta)}{\rho_0 c_0} \sim -\left(\frac{i}{2\pi kr}\right)^{\frac{1}{2}} e^{ikr}[\Omega(\eta)\,Q(\eta)]\,\frac{M(\eta)\sin\theta}{1-M(\eta)\cos\theta} \times\left[\frac{\kappa_+(k/M(\eta))}{\kappa_+(k\cos\theta)}\,\frac{C(k\cos\theta)}{V_o(k\cos\theta,0+)}\right] \quad \text{as} \quad kr\to\infty, \tag{6.11}$$

where we have introduced the cylindrical co-ordinates $r = (x^2+y^2)^{\frac{1}{2}}$ and $\theta = \sin^{-1} y/r$. The terms involving r are just the usual multiplicative factors that occur in the far-field pressure owing to any two-dimensional source. It is easy to see from (6.6) and (6.7) that the quantity $\Omega(\eta)\,Q(\eta)$ is simply the amplitude of the upwash velocity induced at the plate by the portion of the incident vorticity that is located at the height η above the plate.

6.3. *Low frequency solution*

In order to interpret the remaining factors in (6.11) it is necessary to solve the ordinary differential equations for p_o and V_o and factorize (6.3) to determine κ_+. We can obtain simple explicit formulae by restricting our attention to the long-wavelength limit $k\delta \ll 1$ (where δ is the characteristic width of the shear layer) and supposing that η is in the near field, i.e. $\eta = O(\delta)$. Then it is easy to verify that when $y = O(\delta)$ the asymptotic solutions P_1, V_1 and U_1 of (3.2) that have the behaviour (3.4) are

$$\left.\begin{aligned} \frac{P_1(\alpha,y)}{\rho_0 c_0} &= C_1\,k\int_{\eta^+(\alpha)}^{y}(1-\kappa M)^2\,dy+O(k^2),\\ V_1(\alpha,y) &= (C_1/i)\,(1-\kappa M)+O(k^2),\\ U_1(\alpha,y) &= (-C_1/k)\,M'+O(k),\end{aligned}\right\} \tag{6.12}$$

where $\kappa \equiv \alpha/k = O(1)$ and the function η^+ is defined immediately above (3.25). On the other hand, it follows from (3.8) that the outgoing-wave solution $P_o(\alpha,y)$ must be of the form

$$\frac{P_o(\alpha,y)}{\rho_0 c_0} = \tilde{C}_0 + kB_0 + (A_0 + kA_1)\int(1-\kappa M)^2\,dy + O(k^2)$$

for $y = O(\delta)$ while for $ky = O(1)$ its asymptotic representation is

$$\frac{P_o(\alpha,y)}{\rho_0 c_0} = C_0\exp\{-[\kappa^2-(1-\kappa M_\infty)^2]^{\frac{1}{2}}\,ky\}+O(k) \quad \text{for} \quad ky = O(1). \tag{6.13}$$

Matching these inner and outer expansions in the overlap region (see Cole 1968, pp. 7–13) we find that $\tilde{C}_0 = C_0$, $A_0 = 0$ and $A_1 = -C_0[\kappa^2-(1-\kappa M_\infty)^2]^{\frac{1}{2}}/(1-\kappa M_\infty)^2$. And, since we have set $M_\infty = 0$, it follows from $(3.2a)$ that

$$P_o(\alpha,y)/\rho_0 c_0 = C_0 + O(k)$$

and

$$V_o(\alpha,y) = iC_0(\kappa^2-1)^{\frac{1}{2}}\,(1-\kappa M)+O(k)$$

when $y = O(\delta)$.

Hence it follows from (6.3) that

$$\kappa_-(\alpha)/\kappa_+(\alpha) = i(\alpha-k)^{\frac{1}{2}}\,(\alpha+k)^{\frac{1}{2}}\,(k-\alpha M_0)/\rho_0 c_0 k^2,$$

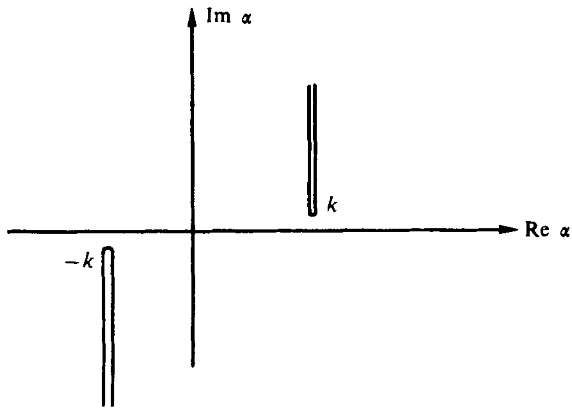

FIGURE 4. Branch cuts for square root in complex α plane.

where the branch cut of the square root is as shown in figure 4. Consequently

$$\kappa_+(\alpha) = (\alpha + k)^{-\frac{1}{2}},$$

and, since comparison of (6.13) and (6.10) shows that $C(\alpha) = C_0$, (6.11) becomes

$$\frac{\overline{p}(x,y|\eta)}{\rho_0 c_0} \sim -\left(\frac{i}{2\pi r k}\right)^{\frac{1}{2}} e^{ikr} M(\eta)\,[\Omega(\eta)\,Q(\eta)] \times \frac{2^{\frac{1}{2}} \cos\frac{1}{2}\theta\,\{M(\eta)/[1+M(\eta)]\}^{\frac{1}{2}}}{[1-M(\eta)\cos\theta]\,(1-M_0\cos\theta)} \quad \text{as} \quad kr \to \infty, \quad \delta k \to 0, \tag{6.14}$$

where in view of (6.7)

$$Q(\eta) \sim ik(M(\eta) - M_0)/M(\eta)\,M'(\eta) \quad \text{as} \quad \delta k \to 0, \quad \eta = O(M'/M).$$

6.4. *Interpretation of results*

The fact that the source is embedded in a moving medium is reflected by the appearance of the two Doppler factors $1 - M(\eta)\cos\theta$ and $1 - M_0\cos\theta$. The first of these depends on the mean flow Mach number at the vertical position of the gust, which can, to a first approximation, be thought of as the convection velocity of the gust. The second depends on the Mach number at the position of the plate. The remaining part of the directivity pattern is determined by the factor $\cos\frac{1}{2}\theta$.

It is instructive to compare (6.14) with the sound field produced by a non-compact free-space dipole in a non-moving medium whose strength is given by the pressure fluctuations that would be produced at the plate if it were placed in a uniform flow and subjected to a convected gust. This is the type of model that one might arrive at if one attempted to solve the problem of this section by using the 'acoustic analogy' approach (Lighthill 1952; Curle 1955). It involves the assumption that the gust will generate surface pressure fluctuations as if it were embedded in a completely uniform flow and that these surface pressure fluctuations will then generate the same sound field as they would produce in a stationary medium. Thus the amplitude of the pressure fluctuation at the point (x, y) due to the stationary medium acoustic dipole produced by a two-dimensional harmonic surface pressure fluctuation $\bar{p}(x,y)\,e^{-i\omega t}$ acting on the plate is

$$\bar{p}(x,y) = -\frac{1}{2\pi}\frac{\partial}{\partial y}\int_0^\infty \bar{p}(x',0)\int_{-\infty}^{\infty}\frac{e^{ikr}}{R}\,dz'\,dx',$$

where

$$R = [(x-x')^2+y^2+(z-z')^2]^{\frac{1}{2}}.$$

The inner integral can be expressed in terms of a Hankel function, which can be expanded for large kr to obtain

$$\bar{p}(x,y) \sim -\frac{1}{2\pi}\left(\frac{2\pi i}{kr}\right)^{\frac{1}{2}} ik\sin\theta\int_0^\infty \bar{p}(x',0)\exp\left[ik(r-x'\cos\theta)\right]dx' \quad \text{as} \quad kr\to\infty. \tag{6.15}$$

On the other hand, the amplitude of the surface pressure fluctuations that would be produced if the plate were placed in a uniform flow with Mach number M_e and subjected to a frozen harmonic gust $v_0\exp\{ik[(x/M_e)-c_0t]\}$ is (Goldstein 1976, p. 138)

$$\frac{\bar{p}(x,0)}{\rho_0 c_0} = -\frac{iM_e v_0\exp\left[ikx/(1+M_e)\right]}{[\pi i(1+M_e)\,kx/M_e]^{\frac{1}{2}}}.$$

Inserting this into (6.15) and carrying out the integration yields

$$\frac{\bar{p}(x,y)}{\rho_0 c_0} \sim -\left(\frac{i}{2\pi kr}\right)^{\frac{1}{2}} e^{ikr}M_e v_0\frac{(\sin\theta)\,M_e^{\frac{1}{2}}}{[1-(1+M_e)\cos\theta]^{\frac{1}{2}}}.$$

Since ΩQ is the upwash velocity amplitude at the plate associated with the gust and since $(1-\cos\theta)^{-\frac{1}{2}} = 2^{\frac{1}{2}}\cos\frac{1}{2}\theta/\sin\theta$, it is clear that this result agrees with (6.14) as the mean flow Mach number becomes small. (Of course, for reasons given in § 3 we cannot expect the relation between the upwash velocity and incident vorticity to be the same.) For finite Mach numbers the convection effects associated with the gust and the mean flow cause the two results to differ.

7. Concluding remarks

The general behaviour of an unsteady gust in a transversely sheared mean flow is studied and its connexion with the frozen disturbances that occur on a uniform flow is pointed out. The general ideas are illustrated by considering the scattering of a gust by a semi-infinite plate in a non-uniform mean flow. Simple formulae are obtained for the far-field behaviour of the resulting acoustic radiation. They are found to exhibit some interesting convective effects.

Appendix A. Representation of solutions for symmetric velocity profiles

In this appendix we show that the solution (3.3) can be written in the form (3.7) when $M(-y) = M(y)$. Thus, if $P(\alpha,y)$, $V(\alpha,y)$ and $U(\alpha,y)$ satisfy (3.2), $P(\alpha,-y)$, $-V(\alpha,-y)$ and $U(\alpha,-y)$ will also be solutions to these equations, and, since P_1, V_1 and U_1 contain no logarithmic terms, neither will $P_1(\alpha,-y)$, $-V_1(\alpha,-y)$ and $U_1(\alpha,-y)$. Then since any solution can be expressed as a linear combination of any two linearly independent solutions and since $\{P_1(\alpha,-y), -V_1(\alpha,-y), U_1(\alpha,-y)\}$ is a solution which contains no logarithmic terms, it follows that it must be equal to a constant multiple of $\mathbf{Z}_1$. But, since $\mathbf{Z}_1$ is only defined to within a constant multiple, it follows that we can write

$$\mathbf{Z}_1(\alpha, y) = \{P_1(\alpha, |y|),\ (\operatorname{sgn} y)\, V_1(\alpha, |y|),\ U_1(\alpha, |y|)\}.$$

Appendix B. Determination of condition for uniform approach to constant velocity limit

In this appendix we establish a condition that will ensure that Ω_e will approach $\Omega(y)$ as M becomes uniform. It was shown in § 4 that this will occur if

$$\Pi(k/M(\eta), y) \equiv (ik/M(\eta))\, V(k/M(\eta), y) - U'(k/M(\eta), y) \tag{B 1}$$

approaches zero as M becomes uniform. In order to determine when this will happen, notice that the argument used to obtain (4.7) from (4.6) also implies that

$$U'(k/M(\eta), y) \approx (iM/k)\, V''(k/M(\eta), y)$$

as M becomes constant. Consequently

$$\Pi(k/M(\eta), y) \approx \frac{iM}{k}\left[\left(\frac{k}{M}\right)^2 V(k/M, y) - V''(k/M, y)\right] \tag{B 2}$$

as M becomes constant. This is essentially equivalent to requiring that $V(\alpha, y)$ approaches a solution of the constant mean velocity reduced wave equation

$$V'' + [(k-\alpha M)^2 - \alpha^2]\, V = 0$$

uniformly in α as M becomes constant. However, this will not usually happen. Thus elimination of P and U between (3.2*a–c*) shows that $V(\alpha, y)$ satisfies the equation

$$\left[\frac{\alpha M' V + (k-\alpha M)\, V'}{(k-\alpha M)^2 - \alpha^2}\right]' + (k-\alpha M)\, V = 0. \tag{B 3}$$

Consequently $V(k/M(\eta), y)$ satisfies the equation

$$\Delta M\, V'' + \frac{2M'(\Delta M)^2}{(\Delta M)^2 - 1} V' + \left\{M'' + \frac{2M'^2 \Delta M}{(\Delta M)^2 - 1} + \Delta M \left[\frac{k}{M(\eta)}\right]^2 [(\Delta M)^2 - 1]\right\} V = 0, \tag{B 4}$$

where $\Delta M \equiv M(\eta) - M(y)$. As M becomes uniform, M', ΔM, $M'' \to 0$ and this becomes

$$V'' - \frac{k^2}{M^2(\eta)}\left[1 - \frac{M''}{\Delta M}\frac{M^2(\eta)}{k^2}\right] V = 0. \tag{B 5}$$

The right side of (B 2) will therefore not vanish unless $|M'' M^2(\eta)/\Delta M k^2| \ll 1$. This inequality can certainly not be satisfied at every point since $\Delta M \equiv 0$ when $y = \eta$ whereas $M''(\eta)$ will generally be finite there. But, for y close to η, $M''(y) \approx M''(\eta)$ and $\Delta M \approx M'(\eta)(\eta - y)$ if $M'(\eta) \neq 0$. Thus, if

$$\epsilon \equiv \frac{M''(\eta)}{M'(\eta)}\frac{M(\eta)}{k}, \quad |\epsilon| \ll 1, \tag{B 6}$$

then

$$M'' M^2(\eta)/\Delta M k \approx -\epsilon/Z,$$

where

$$Z \equiv (y-\eta)\, k/M(\eta), \tag{B 7}$$

and (B 5) becomes

$$d^2V/dZ^2-(1-\epsilon/Z)\,V=0. \tag{B 8}$$

For $Z=O(1)$ this equation possesses the solution

$$V=C_1\sinh Z+C_2\cosh Z,$$

while for small values of Z we can seek a solution in the usual way (Cole 1968, pp. 7–13) by introducing the inner variable $\hat{Z}\equiv Z/\beta(\epsilon)$, $\beta=O(\epsilon)$, to obtain

$$d^2\hat{V}/d\hat{Z}^2+\beta(\epsilon/\hat{Z}-\beta)\,\hat{V}=0.$$

But since the second term will be negligible compared with the first for every allowable choice of β, the inner solution must be $\hat{V}=a_1+a_2\hat{Z}$. We must therefore take $a_1=0$ in the solution V_1 that vanishes at $y=\eta$, while matching the inner and outer solutions shows that $a_2=\beta C_1$. Consequently,

$$V_1(k/M(\eta),y)=C_1\operatorname{sgn}y\sinh|Z| \tag{B 9}$$

is a uniformly valid solution to (B 4) in the range $0\leqslant Z\leqslant O(1)$ as $\Delta M,\epsilon\to 0$. And, since ΔM is required to vanish, this solution is also valid for $Z\gg 1$.

Similarly we can show that

$$V_o(k/M(\eta),y)=C_o\operatorname{sgn}y\exp(-|y|k/M) \tag{B 10}$$

is a uniformly valid outgoing-wave solution to (B 4). Thus both $\Pi_1(k/M(\eta),y)$ and $\Pi_o(k/M(\eta),y)$ will go to zero and Ω_e will approach Ω as $\Delta M\to 0$, provided that we also require that $\epsilon\to 0$. Consequently, Ω_e will represent the true upstream vorticity as M becomes constant if we require that the radius of curvature of the mean velocity profile simultaneously becomes large relative to a wavelength in the sense dictated by (B 6).

It is instructive to show that, at least in this limit, (3.30) will indeed reduce directly to (2.4a) with $\hat{V}$ given by (2.7). This will be accomplished if we can show that the second component of (4.3) reduces to (2.7). For simplicity we consider the case of a symmetric velocity profile. Then on inserting (B 9) and (B 10) into (4.7) we find $U_1(k/M(\eta),y)\sim iC_1\cosh|Z|$ and $U_o(k/M(\eta),y)\sim -iC_o\exp(-k|y|/M)$. Inserting these results into the symmetric velocity profile formula (3.2) yields

$$\begin{aligned}\gamma(\eta)&\sim\frac{-C_1}{2C_o}\left[\cosh\frac{k}{M}|\eta|+\operatorname{sgn}\eta\sinh\frac{k}{M}|\eta|\right]=-\frac{C_1}{2C_o}\exp[(k/M)|\eta|\operatorname{sgn}\eta]\\&=-\frac{C_1}{2C_o}\exp[(k/M)\,\eta].\end{aligned}$$

Then inserting this together with the appropriate formulae for U and V into the second component of (4.3) shows that this expression does indeed reduce to (2.7).

REFERENCES

Betchov, R. & Criminale, W. O. 1967 *Stability of Parallel Flows.* Academic Press.

Carrier, G. F. & Carlson, F. D. 1946 *Quart. Appl. Math.* **4**, 1–12.

Carrier, G. F., Krook, M. & Pearson, C. E. 1966 *Functions of a Complex Variable.* McGraw-Hill.

Case, K. M. 1960 Stability of an idealized atmosphere, I. Discussion of results. *Phys. Fluids* **3**, 149–154.

Chu, B. J. & Kovásznay, L. S. G. 1958 Non-linear interactions in a viscous heat conducting gas. *J. Fluid Mech.* **3**, 494–515.

Cole, J. D. 1968 *Perturbation Methods in Applied Mathematics*. Blaisdell.

Curle, N. 1955 The influence of solid boundaries on aerodynamic sound. *Proc. Roy. Soc.* A **231**, 505–514.

Goldstein, M. E. 1976 *Aeroacoustics*. McGraw-Hill.

Kovásznay, L. S. G. 1953 Turbulence in supersonic flow. *J. Aero. Sci.* **20**, 657–674.

Lighthill, M. J. 1952 On sound generated aerodynamically, I. General theory. *Proc. Roy. Soc.* A **211**, 564–587.

Möhring, W. 1976 Über Schallwellen in Scherströmungen. *Fortschritte der Akustik*, no. DAGA **76**, pp. 543–546. VDI–Verlag.

Noble, B. 1958 *Methods Based on the Weiner–Hopf Technique*. Pergamon.

Phillips, O. M. 1960 On the generation of sound by supersonic turbulent shear layers. *J. Fluid Mech.* **9**, 1.

Stakgold, I. 1967 *Boundary Value Problems in Mathematical Physics*, vol. 1, p. 53. Macmillan.

Woods, L. C. 1961 The Plemelj formulae. *The Theory of Subsonic Plane Flow*, pp. 82–84. Cambridge University Press.